Resistive Ballooning Modes in Three-Dimensional Configurations

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To Professor Arnulf Schlüter on his 60th Birthday

Resistive ballooning modes in general three-dimensional configurations are studied on the basis of the equations of motion of resistive MHD. Assuming small, constant resistivity and perturbations localized transversally to the magnetic field, a stability criterion is derived in the form of a coupled system of two second-order differential equations. This criterion contains several limiting cases, in particular the ideal ballooning mode criterion and criteria for the stability of symmetric systems. Assuming small growth rates, analytical results are derived by multiple-length-scale expansion techniques. Instabilities are found, their growth rates scaling as fractional powers of the resistivity.

I. Introduction

The purpose of this paper is to derive a criterion for the stability of an arbitrary three-dimensional toroidal plasma with respect to resistive ballooning instabilities. The basic restrictions imposed to make the problem tractable are the assumptions of small constant resistivity and localized perturbations. As in the ideal case [1], the introduction of localized perturbations makes it possible to reduce the calculations to the neighborhood of any particular, closed field line. The two coordinates which define this field line then only enter the problem as parameters and the calculations are reduced to a one-dimensional problem along each closed field line. Even with the considerable simplification thus achieved, the stability criterion obtained is, in general, rather implicit since its evaluation requires the solution of a system of two coupled second-order ordinary differential equations on each closed field line. However, assuming small growth rates, it is possible to obtain general, analytical results. In particular, the criterion contains several limiting cases, e.g. the ideal ballooning mode criterion [1] (and thus Mercier's criterion) and the condition $D_R > 0$ [2] for instability with respect to resistive interchanges.

In Sect. II we introduce the model and derive the resistive ballooning mode equations. In Sect. III we study these equations on the assumption of small growth rates, making use of multiple-length-scale

expansion techniques and derive a dispersion relation for the growth rate. In Sect. IV we discuss this dispersion relation and in Sect. V we summarize the results.

II. Resistive Ballooning Mode Equations

We start with the following linearized equations of resistive MHD [3]:

$$\varrho \gamma^2 \boldsymbol{\xi} = - \nabla (\tilde{p} + \delta \boldsymbol{B} \cdot \boldsymbol{B}) + (\boldsymbol{B} \cdot \nabla) \delta \boldsymbol{B} + (\delta \boldsymbol{B} \cdot \nabla) \boldsymbol{B}, \tag{1}$$

$$\delta \boldsymbol{B} = \nabla \times \boldsymbol{\xi} \times \boldsymbol{B} + (\eta/\gamma) \varDelta \delta \boldsymbol{B}, \tag{2}$$

$$\tilde{p} = -\boldsymbol{\xi} \cdot \nabla p - \gamma_H p \nabla \cdot \boldsymbol{\xi}, \tag{3}$$

which describe small-amplitude perturbations $\boldsymbol{\xi}$ (fluid displacement), $\delta \boldsymbol{B}$ (perturbed magnetic field) and $\tilde{p}$ (perturbed pressure) around an equilibrium with scalar pressure p, density ϱ, magnetic field $\boldsymbol{B}$ and small constant resistivity η (the smallness of the resistivity to be specified later). Here γ_H is the ratio of the specific heats. A time dependence of the form $\boldsymbol{\xi}(\boldsymbol{r}, t) = \boldsymbol{\xi}(\boldsymbol{r}) e^{\gamma t}$ has been assumed. Terms associated with the diffusion velocity due to η have been neglected since they have no influence on the instabilities to be considered here (these having a much larger growth rate than ordinary resistive diffusion). On the basis of the lowest-order equilibrium (in an η-expansion), which is taken to be a static, ideal MHD equilibrium, we introduce coordinates $v, \theta, \varphi = \zeta - q_0 \theta$, where v, θ, ζ are Hamada coordinates [4] and $q_0 = M/N$ (M, N integers) is the safety factor of a reference rational surface $v = v_0$ [1]. In these coordinates, the physical quantities

satisfy the following periodicity conditions:

$$\Phi(\theta, \varphi) = \Phi(\theta, \varphi + 1)$$
$$= \Phi(\theta + 1, \varphi - M/N) \qquad (4)$$
$$= \Phi(\theta + N, \varphi),$$

and are thus periodic in θ with period N, and in φ with period 1. This follows from the definition of φ and from the fact that — in Hamada coordinates — all quantities are periodic in both θ and ζ with period 1.

The equilibrium magnetic field $\boldsymbol{B}$ and the gradient along a field line can be expressed as

$$\boldsymbol{B} = \dot{\chi}[\nabla\varphi \times \nabla v + (q - q_0)\nabla v \times \nabla\theta], \quad (5)$$

$$\boldsymbol{B} \cdot \nabla = \dot{\chi}[\partial_\theta + (q - q_0)\partial_\varphi], \qquad (6)$$

with $q = \dot{\Psi}/\dot{\chi}$. Ψ and χ are the longitudinal and transverse magnetic fluxes respectively. The derivatives with respect to θ correspond to derivatives along the closed field line $v = v_0$, $\varphi = \varphi_0$. Dots mean derivatives with respect to the volume. It thus follows that $q_0 = \dot{\Psi}/\dot{\chi}$ $(v = v_0)$.

As in the ideal case [1], we look for solutions of Eqs. (1)—(3) which are localized around a closed field line $v = v_0$, $\varphi = \varphi_0$, and which have finite gradients along the field, i.e. $\boldsymbol{B} \cdot \nabla/\dot{\chi} \sim O(1)$.

Setting

$$t = (v - v_0)/v_0\,\varepsilon^3, \quad x = (\varphi - \varphi_0)/\varepsilon^3, \qquad (7)$$

with $\varepsilon \ll 1$, we look for perturbations with $\partial_t \sim \partial_x \sim O(1)$. Derivatives transverse to the field are thus large, being of the order ε^{-3}. Furthermore, the perturbations are required either to vanish or to be negligible for $|v - v_0| \geqq v_0\,\varepsilon^2$ (ie. $|t| \geqq \varepsilon^{-1}$),

$$|\varphi - \varphi_0| \geqq \varepsilon \quad (\text{i.e. } |x| \geqq \varepsilon^{-2}).$$

Taking into account Eq. (2) and the fact that $\nabla \times \nabla \times \sim \varepsilon^{-6}$, we require $\eta \sim O(\varepsilon^6)$ and set

$$\eta = \eta^*\,\varepsilon^6 \equiv \eta^*/k_\perp^2. \qquad (8)$$

This is the aforementioned specification of the smallness of the resistivity. ε is simply a dummy parameter and is treated here as a kind of tag indicating the magnitude of the term of which it is a factor (it will be seen that the instabilities found are proportional to fractional powers of η^*. In comparison, ordinary magnetic diffusion is proportional to $\eta \sim \varepsilon^6$). We now expand the equilibrium quantities

in a Taylor series around the closed field line $v = v_0$, $\varphi = \varphi_0$:

$$A(v, \theta, \varphi) = A(v_0, \theta, \varphi_0) \qquad (9)$$
$$+ \frac{\partial A}{\partial\varphi}(v_0, \theta, \varphi_0)(\varphi - \varphi_0) + O(\varepsilon^2),$$

set

$$\boldsymbol{\xi} = U\nabla\theta \times \nabla\varphi$$
$$+ T\nabla v \times \nabla\theta + S\boldsymbol{B}, \qquad (10)$$

$$\delta\boldsymbol{B} = w\nabla\theta \times \nabla\varphi$$
$$+ \tau\nabla v \times \nabla\theta + \mu\boldsymbol{B} \qquad (11)$$

and expand the perturbations U, T etc. in a series of the form

$$U = U_0 + U_1\varepsilon + U_2\varepsilon^2 + U_3\varepsilon^3 + \cdots, \quad (12)$$

where the order of magnitude of the different terms is given by the powers of ε and the functions U_i, T_i etc. may depend implicitly on ε. Taking into account the condition $\nabla \cdot \delta\boldsymbol{B} = 0$ on the perturbed magnetic field up to terms $O(1)$, we obtain

$$\partial_t W_0 + v_0\,\partial_x\tau_0 = \partial_t W_1 + v_0\,\partial_x\tau_1$$
$$= \partial_t W_2 + v_0\,\partial_x\tau_2$$
$$= \partial_t W_3 + v_0\,\partial_x\tau_3$$
$$+ v_0\boldsymbol{B} \cdot \nabla\mu_0 = 0.$$

We shall only need $\partial_t W_0 + v_0\,\partial_x\tau_0 = 0$, which implies $W_0 = v_0\,\partial_x a$, $\tau_0 = -\partial_t a$. Expanding Eqs. (1) to (3) in ε, we obtain from the $O(\varepsilon^{-3})$, $O(\varepsilon^{-2})$ and $O(\varepsilon^{-1})$ equations $(\tilde{p} + \delta\boldsymbol{B} \cdot \boldsymbol{B})_i = 0$ and $\partial_t U_i + v_0\,\partial_x T_i = 0$, with $i = 0, 1, 2$. We shall only need $(\tilde{p} + \delta\boldsymbol{B} \cdot \boldsymbol{B})_0 = 0$ and $\partial_t U_0 + v_0\,\partial_x T_0 = 0$, i.e. $U_0 = v_0\partial_x\Phi$, $T_0 = -\partial_t\Phi$. We cross Eq. (1) twice with $\boldsymbol{B}$ to obtain $\boldsymbol{\xi}_\perp$, the component of $\boldsymbol{\xi}$ perpendicular to $\boldsymbol{B}$. From the $\nabla\theta \times \nabla\varphi$ and $\nabla v \times \nabla\theta$ components of the resulting $O(1)$ equations we can eliminate $(\tilde{p} + \delta\boldsymbol{B} \cdot \boldsymbol{B})_3$ and obtain

$$\varrho\gamma^2(\dot{\chi}^2/B^2)v_0^2\Delta^*\Phi \qquad (13)$$
$$= 2[v_0\varkappa_v\,\partial_x(\delta\boldsymbol{B} \cdot \boldsymbol{B})_0 - \varkappa_\varphi\,\partial_t(\delta\boldsymbol{B} \cdot \boldsymbol{B})_0]$$
$$+ v_0^2(\boldsymbol{B} \cdot \nabla)^*(\dot{\chi}^2/B^2)\Delta^*a),$$

with

$$\Delta^* = \frac{|\nabla v|^2}{v_0^2}\,\partial_{tt} + 2\frac{\nabla v \cdot \nabla\varphi}{v_0}\,\partial_{tx}$$
$$+ |\nabla\varphi|^2\,\partial_{xx}, \qquad (14)$$

$$(\boldsymbol{B} \cdot \nabla)^* = \dot{\chi}[\partial_\theta + v_0\dot{q}t\,\partial_x]. \qquad (15)$$

Here, the equilibrium quantities only depend on θ, the variable along the localization line (v_0 and φ_0,

which define the closed field line, only entering the calculations as parameters). $\varkappa_v$ and $\varkappa_\varphi$ are covariant components of the curvature $\mathbf{\varkappa} = [(\mathbf{B}/B) \cdot \nabla]\mathbf{B}/B$. More explicitly, they are

$$\varkappa_v = \mathbf{\varkappa} \cdot \nabla\theta \times \nabla\varphi = \frac{1}{2\,\dot{p}}\left[\frac{\dot{p}^2}{B^2} + \ddot{\Psi}\,I - \ddot{\chi}\,J - \dot{q}\,\dot{\chi}^2\,\sigma\right.$$
$$\left. + \dot{p}\,\mathbf{B} \cdot \nabla\left(\frac{\mathbf{B} \cdot \nabla\theta \times \nabla\varphi}{B^2}\right)\right], \quad (16)$$

$$\varkappa_\varphi = \mathbf{\varkappa} \cdot \nabla v \times \nabla\theta = -(\dot{\chi}/2\,\dot{p})\,\mathbf{B} \cdot \nabla\sigma, \quad (17)$$

where J and I are, respectively, the longitudinal and transverse components of the current and $\sigma = \mathbf{j} \cdot \mathbf{B}/B^2$.

Multiplying by either ∇v or $\nabla\varphi$ and integrating the resulting equation, we obtain from the $0(1)$ Eq. (2)

$$a - (\eta^*/\gamma)\Delta^* a = (\mathbf{B} \cdot \nabla)^* \Phi. \quad (18)$$

Multiplying Eqs. (1) and (2) by $\mathbf{B}$, taking into account Eq. (3) and the condition $(\tilde{p} + \delta\mathbf{B} \cdot \mathbf{B})_0 = 0$ and eliminating S_0 yields

$$(\mathbf{B} \cdot \nabla)^*\left(\frac{1}{B^2}[(\mathbf{B} \cdot \nabla)^*(\delta\mathbf{B} \cdot \mathbf{B})_0 - \dot{p}v_0\,\partial_x a]\right)$$
$$- \varrho\,\gamma^2\,\frac{(\gamma_H\,p + B^2)}{\gamma_H\,p\,B^2}[(\delta\mathbf{B} \cdot \mathbf{B})_0 - \dot{p}v_0\,\partial_x\Phi]$$
$$+ \varrho\,\eta^*\gamma\,\frac{1}{B^2}\,\Delta^*(\delta\mathbf{B} \cdot \mathbf{B})_0$$
$$- 2\,\varrho\,\gamma^2(v_0\varkappa_v\,\partial_x\Phi - \varkappa_\varphi\,\partial_t\Phi) = 0. \quad (19)$$

To solve Eqs. (13), (18), (19), we make the ansatz [1]

$$\Phi(t, \theta, x) = f(\varepsilon t, \varepsilon^2 x)\,e^{i\alpha x}$$
$$\cdot \sum_{m=-\infty}^{\infty} \exp\{2\,\pi\,i\,(m/N)\,\theta$$
$$\cdot \int_{-\infty}^{\infty} \exp\{-i\,(2\,\pi\,(m/N) + \alpha\,\dot{q}\,v_0 t)\,y$$
$$\cdot F(y)\,\mathrm{d}y, \quad (20)$$

where f, f_x, f_t vary slowly with both t and x and are assumed either to vanish or to be negligible for $|t| \geqq \varepsilon^{-1}$, $|x| \geqq \varepsilon^{-2}$. $\alpha \sim 0(1)$ is an arbitrary constant and $F \in L_2$ in $-\infty < y < \infty$. (Representations of periodic functions through functions $F \in L_2(-\infty, \infty)$ are treated in standard books on the theory of approximation of functions [5], [6], [7].) For a and $(\delta\mathbf{B} \cdot \mathbf{B})_0$ we make a similar ansatz

with transformation variables $\tilde{a}(y)$ and $b(y) \in L_2$ $(-\infty, \infty)$ respectively. Multiplying Eqs. (13), (18) and (19) by $f^* \exp\{i\alpha(v_0\dot{q}t\theta - x)\}$ (f^* is the complex conjugate of f), integrating with respect to t between $-\varepsilon^{-1}$ and ε^{-1} and with respect to x between $-\varepsilon^{-2}$ and ε^{-2}, and taking into account that

$$\sum_{m=-\infty}^{\infty} \exp\{2\,\pi\,i\,(m/N)(\theta - y)\}$$
$$= N \sum_{n=-\infty}^{\infty} \delta(\theta - y - nN),$$

with δ the δ-function, and that

$$A(\theta) \cdot \Phi(\theta) = f e^{i\alpha x} \sum_{m=-\infty}^{\infty} \exp\{2\,\pi\,i\,(m/N)\,\theta\}$$
$$\cdot \int_{-\infty}^{\infty} \exp\{-i\,(2\,\pi\,(m/N) + \alpha\,\dot{q}\,v_0 t)\,y\}$$
$$\cdot A(y)\,F(y)\,\mathrm{d}y, \quad (21)$$

with Φ from Eq. (20) and A a periodic function of its argument (period N), one obtains (to lowest order in ε)

$$\varrho\,\gamma^2\,\frac{\dot{\chi}^2}{B^2}\,\mathbf{C}^2 F = -\frac{2\,i}{v_0\alpha}(\varkappa_v + \varkappa_\varphi\,\dot{q}\,y)\,b$$
$$+ \frac{\mathrm{d}}{\mathrm{d}y}\frac{\dot{\chi}^3}{B^2}\,\mathbf{C}^2\,\tilde{a}, \quad (22)$$

$$(1 + (\alpha^2\eta^*/\gamma)\,\mathbf{C}^2)\,\tilde{a} = \dot{\chi}\,(\mathrm{d}F/\mathrm{d}y), \quad (23)$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\frac{\dot{\chi}^2}{B^2}\left[\frac{\mathrm{d}b}{\mathrm{d}y} - \frac{i\alpha}{\dot{\chi}}\,v_0\,\dot{p}\,\tilde{a}\right]$$
$$- \varrho\,\gamma^2\,\frac{(\gamma_H\,p + B^2)}{\gamma_H\,p\,B^2}\,(b - i\alpha\,v_0\,\dot{p}F)$$
$$- \frac{\alpha^2\eta^*}{B^2}\,\gamma\,\varrho\,\mathbf{C}^2\,b \quad (24)$$
$$- 2\,i\alpha\,v_0\,\varrho\,\gamma^2(\varkappa_v + \varkappa_\varphi\,\dot{q}\,y)\,F = 0,$$

with

$$\mathbf{C} = \nabla\varphi - \dot{q}\,y\,\nabla v. \quad (25)$$

Solving for $\tilde{a}$ from (23) and substituting in (22) and (24), multiplying (22) by $i\alpha(\dot{p}v_0/\dot{\chi}^2) \cdot (\alpha^2\eta^*/\gamma)$, substracting this from (24) and setting

$$b = i\alpha\,\dot{p}\,v_0(F + D), \quad (26)$$

we finally obtain the following resistive ballooning mode equations (valid in any geometry and not

restricted to symmetric configurations):

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{\boldsymbol{C}^2}{B^2\left(1+(\alpha^2\eta^*/\gamma)\,\boldsymbol{C}^2\right)}\frac{\mathrm{d}F}{\mathrm{d}y}\right]$$
$$+\frac{2\dot{p}}{\dot{\chi}^4}\left(\varkappa_v+\dot{q}\,y\,\varkappa_\varphi\right)F-\frac{\varrho\,\gamma^2}{\dot{\chi}^2\,B^2}\,\boldsymbol{C}^2F \quad (27)$$
$$=-\frac{2\dot{p}}{\dot{\chi}^4}\left(\varkappa_v+\dot{q}\,y\,\varkappa_\varphi\right)D,$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{1}{B^2}\frac{\mathrm{d}D}{\mathrm{d}y}\right]-\frac{\varrho\,\gamma^2}{\dot{\chi}^2}\frac{(\gamma_{\mathrm{H}}\,p+B^2)}{\gamma_{\mathrm{H}}\,p\,B^2}\,D$$
$$-\frac{\varrho\,\alpha^2}{\dot{\chi}^2\,B^2}\,\eta^*\,\gamma\,\boldsymbol{C}^2D-\frac{2\dot{p}}{\dot{\chi}^4}\frac{\alpha^2\,\eta^*}{\gamma}\left[\varkappa_v+\dot{q}\,y\,\varkappa_\varphi\right]D$$
$$\hspace{4cm}(28)$$
$$=\frac{2\varrho}{\dot{p}\,\dot{\chi}^2}\left(\frac{\dot{p}^2\,\alpha^2\,\eta^*}{\dot{\chi}^2\,\gamma\,\varrho}+\gamma^2\right)\cdot\left(\varkappa_v+\dot{q}\,y\,\varkappa_\varphi\right)F.$$

Here, the differentiation with respect to y corresponds to a differentiation along the localization line: $\dot{\chi}\,(\mathrm{d}/\mathrm{d}y)=(\boldsymbol{B}\cdot\nabla)\,(v=v_0,\varphi=\varphi_0)$. Besides the equilibrium quantities, these equations only contain F (related to the normal component of the displacement $\boldsymbol{\xi}$) and D (related to $\nabla\cdot\boldsymbol{\xi}$). The equilibrium quantities depend on the particular localization line through the parameters v_0 and φ_0 (Eq. (9)) and have an explicit periodic dependence (period N) on the variable y along this line. Our stability criterion is thus as follows: the system is unstable with respect to resistive ballooning modes if there are solutions $F, D \in L_2(-\infty, \infty)$ with $\Re e\,\gamma > 0$, with $\Re e\,\gamma$ the real part of the growth rate.

III. Analysis of the Resistive Ballooning Mode Equations

Equations (27) and (28) govern the localized resistive ballooning modes and contain several particular cases:

III. A) $\eta^*/\gamma = \gamma = 0$ (ideal marginal stability)

If we set $\eta^*/\gamma = \gamma = 0$, we obtain the ideal marginally stable case. This has been treated elsewhere [1, 8, 9, 10, 11]. Here we assume that the equilibrium is stable in the ideal case. By taking resistivity into account we then introduce the possibility of new instabilities, as in [2, 12, 13, 14].

III. B) $\gamma_{\mathrm{H}} = 0$

If we assume that the perturbed pressure (Eq. (3)) is determined by convection alone and neglect the effect of the compressibility term by setting $\gamma_{\mathrm{H}} = 0$, (27) and (28) then reduce to $D = 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{1}{B^2}\frac{\boldsymbol{C}^2}{(1+(\alpha^2\eta^*/\gamma)\,\boldsymbol{C}^2)}\frac{\mathrm{d}F}{\mathrm{d}y}\right] \quad (29)$$
$$+\frac{2\dot{p}}{\dot{\chi}^4}\left(\varkappa_v+\dot{q}\,y\,\varkappa_\varphi\right)F-\frac{\varrho\,\gamma^2}{\dot{\chi}^2\,B^2}\,\boldsymbol{C}^2F=0.$$

For a given equilibrium, (29) can be solved numerically. Some results can also be obtained analytically. This is done in the following.

In axisymmetric systems, (29) is essentially the same equation as was derived in [15] for the case of ballooning modes with high toroidal mode number. However, the validity of (29) is not restricted to the symmetric case. Furthermore, contrary to [15], we do not find here in general instabilities with growth rates proportional to the resistivity. The results obtained there are due to the fact that the dependence of the integrals in (20), (21) of [15] on both γ and η was ignored. Comparing (29) with the results obtained in the ideal case [1], it is clear that the effect of resistivity is to reduce the stabilizing contribution of field line bending — which is represented by the first term in (29) — by the factor $(1 + (\alpha^2\eta^*/\gamma)\,\boldsymbol{C}^2)^{-1}$.

III. C) $\gamma_{\mathrm{H}} = 0$, $(\alpha^2\eta^*/\gamma)\,\boldsymbol{C}^2 \gg 1$ and cylindrical symmetry

It is elucidating and easier to consider first the cylindrically symmetric case. Then, $\varkappa_\varphi = 0$, $\nabla v\cdot\nabla\varphi = 0$ and the equilibrium quantities are independent of y. We assume that the growth rate γ is real and small (in all the following cases we assume γ to be real, this being consistent with the solutions found) and set $\boldsymbol{C}^2/(1+(\alpha^2\eta^*/\gamma)\,\boldsymbol{C}^2) \approx \gamma/\alpha^2\eta^*$, neglecting terms of $O(\gamma^2)$ or higher (the results will be seen to be consistent with the assumption of small γ for sufficiently small driving term $\dot{p}\varkappa_v$). We also neglect the term $\gamma^2|\nabla\varphi|^2$ in $\gamma^2\boldsymbol{C}^2$ but retain $\gamma^2y^2|\nabla v|^2$ since this can become large even for small γ. Equation (29) then reduces to

$$\mathrm{d}^2F/\mathrm{d}z^2 + (\varLambda - z^2)F = 0, \quad (30)$$
$$\varLambda = 2\dot{p}\varkappa_v B^2\,|\alpha/\dot{q}|\,|\nabla v|\,\dot{\chi}^3|\,(\eta^*/\varrho\,\gamma^3)^{1/2}, \quad (31)$$
$$z^2 = |\alpha\,\dot{q}|\,|\nabla v|/\dot{\chi}|\,(\varrho\,\gamma\,\eta^*)^{1/2}\,y^2, \quad (32)$$

to be solved with the condition $F \in L_2(-\infty, \infty)$. For $\dot{p}\varkappa_v > 0$ (convex field lines, $\dot{p} < 0$), the solu-

tions of (30) are given by

$$F_n = \exp\{-z^2/2\} H_n, \tag{33}$$

$$\gamma_n^3 = (\eta^*/\varrho)(2\,\dot{p}\varkappa_v\,\alpha\,B^2/\dot{q}\,\dot{\chi}^3\,|\nabla v|)^2\,\frac{1}{(1+2\,n)^2},$$
$$n = 0, 1, \ldots, \tag{34}$$

with H_n the Hermite polynomial of order n. There is thus an infinite sequence of unstable modes, the fastest growing one being obtained for $n = 0$.

The condition for instability ($\dot{p}\varkappa_v > 0$) is the same as in the case of resistive interchanges [3] and is obviously more stringent than Suydam's condition

$$2\,\dot{p}\varkappa_v \geqq \frac{1}{4}\,\frac{\dot{q}^2\,\dot{\chi}^4\,\oint \mathrm{d}y}{\oint (B^2/|\nabla v|^2)\,\mathrm{d}y}$$
[16].

III. D) $\gamma_H = 0$, $\alpha^2(\eta^*/\gamma)\,C^2 \gg 1$, general geometry (no symmetry required)

We now go back to (29). The results in cylindrical symmetry suggest the scaling

$$\dot{p}\varkappa_v \sim \dot{p}\varkappa_\varphi \sim \gamma^{3/2}.$$

We therefore set

$$\dot{p}\varkappa_v = \overline{\dot{p}\varkappa_v}\,\delta^3, \quad \dot{p}\varkappa_\varphi = \overline{\dot{p}\varkappa_\varphi}\,\delta^3,$$
$$\gamma = \overline{\gamma}\,\delta^2, \quad \delta \ll 1. \tag{35}$$

(The consistency of the scaling must of course be verified by the results.) Like before, δ is simply a tag indicating the magnitude of the term of which it is a factor.

To simplify the notation, we drop the bars. (29) now reads

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{1}{B^2}\left(1 + \frac{\delta^2}{A\,C^2}\right)^{-1}\frac{\mathrm{d}F}{\mathrm{d}y}\right]$$
$$+ \frac{A\,\delta}{\dot{\chi}^4}\left[2\,\dot{p}\varkappa_v - \dot{q}\,\dot{\chi}^2 y\,\frac{\mathrm{d}\sigma}{\mathrm{d}y}\right]F$$
$$- \frac{A\,\gamma^2}{B^2\,\dot{\chi}^2}\,\varrho\,\delta^2 C^2 F = 0. \tag{36}$$

Here we have used (17) and set

$$\alpha^2\,\eta^*/\gamma \equiv A. \tag{37}$$

In order to solve (36), we make use of the two variable expansion procedures described in [17]. For (36) we take y and $z = \delta^{1/2}y$ as the two different

length scales and make the ansatz

$$F(y) = F_0(y, z) + \delta^{1/2} F_1(y, z)$$
$$+ \delta F_2(y, z) + \cdots. \tag{38}$$

$$F_i(y + N, z) = F_i(y, z),$$
$$i = 0, 1, 2, \ldots, \tag{39}$$

$$\frac{\mathrm{d}F}{\mathrm{d}y} = \frac{\partial F}{\partial y} + \delta^{1/2}\frac{\partial F}{\partial z}, \tag{40}$$

and solve (36) order by order.

From the lowest-order equation and condition (39) we obtain

$$F_0 = F_0(z). \tag{41}$$

Thus F_0 does not depend explicitly on y. From the next order and (39) it follows that

$$F_1(y, z) = f_1(z) + \frac{A\,\dot{q}\,z}{\dot{\chi}^2}\,F_0\int(\sigma\widetilde{B^2})\,\mathrm{d}y \tag{42}$$
$$+ \frac{1}{\langle B^2\rangle}\left[F_{0z} - \frac{A\,\dot{q}}{\dot{\chi}^2}\langle\sigma B^2\rangle z F_0\right]\int\widetilde{B^2}\,\mathrm{d}y,$$

with f_1 a function of z only.

The brackets have their usual meaning, i.e. they denote mean values on the closed localization line:

$$\langle B^2\rangle = \langle B^2\rangle(v_0, \varphi_0) \tag{43}$$
$$= \frac{\oint B^2\,\mathrm{d}y}{\oint \mathrm{d}y} = \frac{1}{N}\oint B^2\,\mathrm{d}y$$

and

$$\widetilde{B^2} = B^2 - \langle B^2\rangle, \quad \text{i.e.} \quad \langle\widetilde{B^2}\rangle = 0. \tag{44}$$

The integrals in (42) are indefinite integrals without integration constant.

Proceeding to the next order in $\delta^{1/2}$ and taking into account (39), we find a solubility condition for F_2 in the form of a second-order differential equation for $F_0(z)$:

$$\frac{\mathrm{d}^2 F_0}{\mathrm{d}z^2} + \frac{A\langle B^2\rangle}{\dot{\chi}^4}\left[\langle 2\,\dot{p}\varkappa_v\rangle + \dot{q}\,\dot{\chi}^2\right. \tag{45}$$
$$\left.\cdot\left(\langle\sigma\rangle - \frac{\langle\sigma B^2\rangle}{\langle B^2\rangle}\right)\right]F_0 + A^2\dot{q}^2\frac{\langle B^2\rangle}{\dot{\chi}^4}$$
$$\cdot\left[\langle\sigma^2 B^2\rangle - \frac{\langle\sigma B^2\rangle^2}{\langle B^2\rangle} - \frac{\varrho\,\dot{\chi}^2\gamma^2}{A}\left\langle\frac{|\nabla v|^2}{B^2}\right\rangle\right]z^2 F_0 = 0.$$

It is obvious that (30) is a particular case of (45). As in (30), the condition for the existence of solu-

tions $F_0 \in L_2(-\infty, \infty)$ is

$$2 \dot{p} \langle \varkappa_v \rangle + \dot{q} \dot{\chi}^2 \left(\langle \sigma \rangle - \frac{\langle \sigma B^2 \rangle}{\langle B^2 \rangle} \right) > 0 . \qquad (46)$$

There is an infinite sequence of modes with

$$F_{0n} = \exp\{-z^{*2}/2\} H_n(z^*) ,$$
$$n = 0, 1, 2, \ldots , \qquad (47)$$

with H_n the Hermite polynomial of order n and

$$z^* = \left[\frac{\dot{q}^2 A^2 \langle B^2 \rangle}{\dot{\chi}^4} \left\{ \frac{\varrho \gamma^2 \dot{\chi}^2}{A} \left\langle \frac{|\nabla v|^2}{B^2} \right\rangle \right. \right.$$
$$\left. \left. - \langle \sigma^2 B^2 \rangle + \frac{\langle \sigma B^2 \rangle^2}{B^2} \right\} \right]^{1/4} z , \qquad (48)$$

$$\gamma_n^3 = \frac{\alpha^2 \eta^*}{\varrho \dot{\chi}^2 \langle |\nabla v|^2/B^2 \rangle} \left[\langle \sigma^2 B^2 \rangle - \frac{\langle \sigma B^2 \rangle^2}{\langle B^2 \rangle} \right.$$
$$+ \frac{\langle B^2 \rangle}{\dot{q}^2 \dot{\chi}^4 (1 + 2n)^2} \left\{ \langle 2 \dot{p} \varkappa_v \rangle \right.$$
$$\left. \left. + \dot{q} \dot{\chi}^2 \left\{ \langle \sigma \rangle - \frac{\langle \sigma B^2 \rangle}{\langle B^2 \rangle} \right\} \right\}^2 \right] . \qquad (49)$$

The condition (46) for instability is the same as the condition $D_R > 0$ of [2] for configurations with small $\dot{p}$.

III. E) $\gamma_H \neq 0$, $\alpha^2 (\eta^*/\gamma) C^2 \gg 1$

Here, contrary to the preceding case, we drop the requirement $\gamma_H = 0$ in order to study the effect of compressibility (see (3)). For simplicity, we consider first the scaling given by (35), together with

$$p = \bar{p} \delta^3 \qquad (50)$$

and drop the bars to simplify the notation.

We go back to (27) and (28) and make the ansatz (38)—(40) for F. Making a similar ansatz for D and proceeding as in Section II. D, we obtain from the two lowest-order equations

$$F_0 = F_0(z) , \qquad (51)$$

$$D_0 = D_0(z) , \qquad (52)$$

$$F_1(y, z) = f_1(z)$$
$$+ \frac{A}{\dot{\chi}^2} \dot{q} z (F_0 + D_0) \int \left\{ \sigma \widetilde{\widetilde{B^2}} - \frac{\langle \sigma B^2 \rangle}{\langle B^2 \rangle} \widetilde{B^2} \right\} dy$$
$$+ \frac{dF_0}{dz} \frac{1}{\langle B^2 \rangle} \int \widetilde{B^2} dy , \qquad (53)$$

$$D_1(y, z) = d_1(z) - \frac{A}{\dot{\chi}^2} \dot{q} z \left\{ D_0 + F_0 \left(1 + \frac{\varrho \gamma^2 \dot{\chi}^2}{A \dot{p}^2} \right) \right\}$$
$$\cdot \int \left\{ \sigma \widetilde{B^2} - \frac{\langle \sigma B^2 \rangle}{\langle B^2 \rangle} \widetilde{B^2} \right\} dy + \frac{dD_0}{dz} \frac{1}{\langle B^2 \rangle} \int \widetilde{B^2} dy . \qquad (54)$$

Proceeding to the next order in $\delta^{1/2}$, we obtain a solubility condition for F_2 and D_2 in the form of a coupled system of two second-order differential equations for F_0 and D_0

$$\frac{d^2 F_0}{dw^2} + \left(\frac{d_R}{Q^{3/2}} - w^2 \right) F_0 = - \frac{d_R}{Q^{3/2}} D_0 , \qquad (55)$$

$$\frac{d^2 D_0}{dw^2} - \left(\frac{d_R}{Q^{3/2}} + G Q^{3/2} + w^2 \right) D_0$$
$$= \frac{d_R}{Q^{3/2}} (1 + K Q^3) F_0 , \qquad (56)$$

with

$$w = Q^{3/4} x , \qquad (57)$$

$$x = [(\dot{q}^2 A \langle B^2 \rangle)/\langle B^2/|\nabla v|^2 \rangle]^{1/2} z , \qquad (58)$$
$$A = \alpha^2 \eta^*/\gamma ,$$

$$Q^3 = \frac{\varrho \langle B^2/|\nabla v|^2 \rangle M \gamma^3}{\dot{q}^2 \dot{\chi}^2 \langle B^2 \rangle \alpha^2 \eta^*} , \qquad (59)$$

$$M = \langle B^2/|\nabla v|^2 \rangle \qquad (60)$$
$$\cdot \left[\langle |\nabla v|^2/B^2 \rangle + \frac{1}{\dot{p}^2} \left\{ \langle \sigma^2 B^2 \rangle - \frac{\langle \sigma B^2 \rangle^2}{\langle B^2 \rangle} \right\} \right] ,$$

$$d_R = \frac{1}{\dot{q}^2 \dot{\chi}^4} \langle B^2/|\nabla v|^2 \rangle \qquad (61)$$
$$\cdot \left[\langle 2 \dot{p} \varkappa_v \rangle + \dot{q} \dot{\chi}^2 \left\{ \langle \sigma \rangle - \frac{\langle \sigma B^2 \rangle}{\langle B^2 \rangle} \right\} \right] ,$$

$$G = \langle B^2 \rangle / \gamma_H p M , \qquad (62)$$

$$K = \frac{\dot{q}^2 \dot{\chi}^4 \langle B^2 \rangle}{M \dot{p}^2 \langle B^2/|\nabla v|^2 \rangle} . \qquad (63)$$

Taking into account the boundary conditions at $+\infty$, $-\infty$, we obtain the solutions of (55), (56)

$$F_{0n} = e^{-w^2/2} H_n(w) , \qquad n = 0, 1, 2, \ldots , \qquad (64)$$

$$D_{0n} = \left[\frac{(1 + 2n)}{d_R} Q_n^{3/2} - 1 \right] F_{0n} , \qquad (65)$$

$$Q_n^{3/2} = \frac{d_R G - K d_R^2 - (1 + 2n)^2}{G (1 + 2n)} , \qquad (66)$$

with H_n the Hermite polynomial of order n.

The condition for the existence of an instability with mode number n is

$$d_R G - K d_R^2 - (1 + 2n)^2 > 0. \qquad (67)$$

The fastest growing mode is obtained for $n = 0$. If d_R is small, the criterion for instability is

$$d_R > (1 + 2n)^2/G. \qquad (68)$$

This is less stringent than the condition $d_R > 0$ which was obtained in Section II. D with $\gamma_H = 0$, owing to the stabilizing effect of compressibility. For $n = 0$ and large G, the condition for instability is the same as in the case $\gamma_H = 0$. Nevertheless, the growth rates are not the same. This is due to the fact that for $\gamma_H = 0$ one has $D = 0$. For large G, on the other hand, only D_0 (the lowest-order term in the $\delta^{1/2}$ expansion) vanishes.

The cases studied under II.C)−II.E) require a small driving term in order that the condition $(\alpha^2 \eta^*/\gamma) C^2 \gg 1$ be satisfied. We now drop this assumption and consider:

III.F) $\gamma_H \neq 0$, $\eta^*/\gamma \sim \gamma^2$

In this case, resistivity and inertia are equally important. We now set

$$\eta^*/\gamma = \overline{(\eta^*/\gamma)}\,\delta, \quad \gamma^2 = \bar{\gamma}^2\,\delta, \quad \delta \ll 1 \qquad (69)$$

and again drop the bars to simplify the notation. Equations (27) and (28) now read

$$\frac{d}{dy}\left[\frac{1}{B^2}\frac{C^2}{1 + A\,\delta\,C^2}\frac{dF}{dy}\right] \qquad (70)$$
$$+ \frac{1}{\dot{\chi}^4}\left[2\,\dot{p}\,\varkappa_v - \dot{q}\,\dot{\chi}^2 y\,\frac{d\sigma}{dy}\right]F - \frac{\varrho\,\gamma^2}{\dot{\chi}^2 B^2}\,\delta\,C^2 F$$
$$= -\frac{1}{\dot{\chi}^4}\left[2\,\dot{p}\,\varkappa_v - \dot{q}\,\dot{\chi}^2 y\,\frac{d\sigma}{dy}\right]D,$$

$$\frac{d}{dy}\left[\frac{1}{B^2}\frac{dD}{dy}\right] - \frac{\varrho\,\gamma^2\,\delta}{\dot{\chi}^2}\left(\frac{\gamma_H\,p + B^2}{\gamma_H\,p\,B^2}\right)D \qquad (71)$$
$$- \frac{\varrho}{\dot{\chi}^2 B^2}\,A\,\gamma^2\,\delta^2\,C^2 D - \frac{A\,\delta}{\dot{\chi}^4}\left[2\,\dot{p}\,\varkappa_v - \dot{q}\,\dot{\chi}^2 y\,\frac{d\sigma}{dy}\right]D$$
$$= \frac{\varrho\,\delta}{\dot{p}^2\,\dot{\chi}^2}\left(\frac{\dot{p}^2 A}{\varrho\,\dot{\chi}^2} + \gamma^2\right)\left(2\,\dot{p}\,\varkappa_v - \dot{q}\,\dot{\chi}^2 y\,\frac{d\sigma}{dy}\right)F,$$

with $A = \alpha^2\overline{(\eta^*/\gamma)}$.

Since $C^2 = |\nabla\varphi|^2 - 2\,\dot{q}\,y\,\nabla v \cdot \nabla\varphi + \dot{q}^2 y^2\,|\nabla v|^2$, it is clear that resistivity and inertia only play a role for large $|y| \sim \delta^{-1/2}$. There are thus different regions. When $|y| \ll \delta^{1/2}$, resistivity and inertia may

be neglected:

$$\frac{d}{dy}\left[\frac{C^2}{B^2}\frac{dF}{dy}\right] + \frac{1}{\dot{\chi}^4}\left[2\,\dot{p}\,\varkappa_v - \dot{q}\,\dot{\chi}^2 y\,\frac{d\sigma}{dy}\right]F$$
$$= -\frac{1}{\dot{\chi}^4}\left[2\,\dot{p}\,\varkappa_v - \dot{q}\,\dot{\chi}^2 y\,\frac{d\sigma}{dy}\right]D, \qquad (72)$$

$$\frac{d}{dy}\left[\frac{1}{B^2}\frac{dD}{dy}\right] = 0, \quad y \ll \delta^{-1/2}. \qquad (73)$$

Equations (72) and (73) are the same equations as in the ideal marginal case. Setting $D = 0$ (we choose this solution since it will be needed for matching to the resistive regions), it follows that, for large $|y|$, the solution of (72) behaves as

$$F = a_1\,|y|^s + a_2\,|y|^{-1-s}, \quad |y| \to \infty, \qquad (74)$$

where

$$s = -\tfrac{1}{2} + [\tfrac{1}{4} + H^2 - H - D_R]^{1/2} \qquad (75)$$

and H and D_R are defined as in [2], i.e.

$$H = \frac{\langle B^2/|\nabla v|^2\rangle}{\dot{q}\,\dot{\chi}^2}\left[\frac{\langle\sigma B^2\rangle}{\langle B^2\rangle} - \frac{\langle\sigma B^2/|\nabla v|^2\rangle}{\langle B^2/|\nabla v|^2\rangle}\right], \qquad (76)$$

$$D_R = F^* + E + H^2, \qquad (77)$$

$$F^* = \frac{\langle B^2/|\nabla v|^2\rangle}{\dot{q}^2\,\dot{\chi}^4} \qquad (78)$$
$$\cdot\left[\left\langle\frac{B^2\,\sigma^2}{|\nabla v|^2}\right\rangle - \frac{\langle\sigma B^2/|\nabla v|^2\rangle^2}{\langle B^2/|\nabla v|^2\rangle} + \dot{p}^2\left\langle\frac{1}{B^2}\right\rangle\right],$$

$$E = \frac{\langle B^2/|\nabla v|^2\rangle}{\dot{q}^2\,\dot{\chi}^4}\left[I\,\ddot{\Psi} - J\,\ddot{\chi} - \dot{q}\,\dot{\chi}^2\,\frac{\langle\sigma B^2\rangle}{\langle B^2\rangle}\right]. \qquad (79)$$

The equilibria were assumed to be ideally stable. In particular, Mercier's stability criterion is assumed to be satisfied, i.e.

$$-D_I = \tfrac{1}{4} + H^2 - H - D_R \geq 0. \qquad (80)$$

Thus, for large $|y|$ and $-D_I > 0$, the ideal solutions behave as in (74), with the first term predominating. In the outer region ($|y| \gtrsim \delta^{-1/2}$), resistivity and inertia must be taken into account. In order to study this region, we make use of the two-length-scale expansion techniques employed before and make an ansatz similar to (38)−(40) for the functions F and D.

From the two lowest-order equations in the $\delta^{1/2}$ expansion one then obtains

$$F_0 = F_0(z), \quad D_0 = D_0(z), \qquad (81)$$

$$F_1(y,z) = f_1(z) + \left(\frac{dF_0}{dz}\right)\frac{1}{\langle B^2\rangle \dot{q}^2 A z^2 + \langle B^2/|\nabla v|^2\rangle}$$

$$\cdot \int\left\{\left(\widetilde{\frac{B^2}{|\nabla v|^2}}\right) + \dot{q}^2 A z^2 \widetilde{B}^2\right\} dy + \frac{(F_0 + D_0)}{\dot{q}\dot{\chi}^2 z} \quad (82)$$

$$\cdot \int\left\{\left(\widetilde{\frac{B^2}{|\nabla v|^2}}\right) + \left(\widetilde{\frac{B^2\sigma}{|\nabla v|^2}}\right) + A z^2(\widetilde{B}^2 + \widetilde{B^2\sigma})\right\} dy,$$

$$D_1(y,z) = d_1(z) + \frac{dD_0}{dz}\int\frac{\widetilde{B}^2}{\langle B^2\rangle} dy$$

$$+ \dot{q}z\left[\frac{A}{\dot{\chi}^2}(D_0 + F_0) + \frac{\varrho}{\dot{p}^2}\gamma^2 F_0\right] \cdot \quad (83)$$

$$\cdot \int\left(\frac{\langle \sigma B^2\rangle}{\langle B^2\rangle}\widetilde{B}^2 - \widehat{\sigma \widetilde{B}^2}\right) dy.$$

The brackets, tildes and indefinite integrals are already explained in (43) and (44).

To next order, taking into account (39), (58)—(63) and (76)—(79), we obtain a solubility condition for F_2 and D_2 in the form of a coupled system of two second-order ordinary differential equations for F_0 and D_0:

$$\frac{d}{dx}\frac{x^2}{1+x^2}\frac{dF_0}{dx}$$

$$+ \frac{H(1-H)}{(1+x^2)^2}F_0 - \frac{H(1+H)x^2}{(1+x^2)^2}F_0$$

$$+ D_R F_0 - Q^3 x^2 F_0 \quad (84)$$

$$= -H\frac{d}{dx}\left(\frac{x}{1+x^2}D_0\right) + \frac{H^2}{1+x^2}D_0 - D_R D_0,$$

$$\frac{d^2 D_0}{dx^2} + \left[\frac{H^2}{1+x^2} - K Q^3\left(F^* + \frac{x^2}{1+x^2}H^2\right)\right.$$

$$\left. - Q^3 x^2 - G Q^3 - D_R\right]D_0 \quad (85)$$

$$= (1 + K Q^3)\left[D_R F_0 - \frac{H^2}{1+x^2}F_0 - H\frac{x}{1+x^2}\frac{dF_0}{dx}\right],$$

Equations (84) and (85) are independent of the sign of x, it is therefore sufficient to consider positive x values only.

In the following we consider, for simplicity, the case of large G. (As we know from the preceding section, the conclusions drawn in this case are more pessimistic than those obtained with $G \sim 0(1)$.)

In the limit of large G, it follows from (85), that $D_0 = 0$ and one is left with an equation for F_0 alone

$$\frac{d}{dx}\left(\frac{x^2}{1+x^2}\frac{dF_0}{dx}\right) + \frac{H(1-H)}{(1+x^2)^2}F_0 \quad (86)$$

$$- \frac{H(1+H)x^2}{(1+x^2)^2}F_0 + (D_R - Q^3 x^2)F_0 = 0.$$

Equation (86) can be solved exactly. If we set

$$F_0 = |x|^s \exp\{(1+s-H)x^2/2\}\frac{d}{dx^2}$$

$$\cdot [\exp\{(H-1-s-Q^{3/2})x^2/2\}$$

$$\cdot P(r \equiv Q^{3/2}x^2)], \quad (87)$$

with s from (75), one obtains the following equation for $P(r)$

$$r\frac{d^2 P}{dr^2} + \left[\frac{1}{2} + s - r\right]\frac{dP}{dr} \quad (88)$$

$$- \frac{1}{4}\left[Q^{3/2} + 1 + 2s - \frac{D_R}{Q^{3/2}}\right]P = 0,$$

which is Kummer's equation. The solution, which remains finite and thus satisfies the boundary conditions at ∞, is [19]

$$P(r) \sim \frac{\pi}{\sin \pi \nu}\left[\frac{{}_1F_1[a^*; \nu; r]}{\Gamma(1 + a^* - \nu)\Gamma(\nu)}\right. \quad (89)$$

$$\left. - r^{1-\nu}\frac{{}_1F_1[1 + a^* - \nu; 2 - \nu; r]}{\Gamma(a^*)\Gamma(2-\nu)}\right],$$

with Γ the gamma function and

$$\nu = \frac{1}{2} + s,$$
$$a^* = \frac{1}{4}[Q^{3/2} + 2\nu - D_R/Q^{3/2}]. \quad (90)$$

${}_1F_1[a^*; \nu; r]$ is Kummer's function:

$$_1F_1[a^*; \nu; r] \quad (91)$$

$$= 1 + \frac{a^*}{\nu}r + \frac{a^*(a^*+1)}{\nu(\nu+1)}\frac{r^2}{2!} + \cdots.$$

Equation (87) then explicitly reads

$$F_0 = \frac{\pi |x|^s \exp\{-Q^{3/2}x^2/2\}}{\sin \pi \nu \, \Gamma(1 + a^* - \nu)\Gamma(\nu)}\left[\frac{(Q^{3/2} - 1 - s + H)}{2}{}_1F_1(a^*; \nu; r) + \frac{(a^* - \nu)}{\nu}{}_1F_1(a^*; \nu+1; r)\right.$$

$$- \frac{\Gamma(\nu)\Gamma(1 + a^* - \nu)}{\Gamma(1 - \nu)\Gamma(a^*)}Q^{\frac{3}{2}(\frac{1}{2}-s)}|x|^{-1-2s}\left\{{}_1F_1(a^* - \nu; 1 - \nu; r)\right. \quad (92)$$

$$\left.\left. + \frac{(Q^{3/2} - 1 - s + H)}{2(1-\nu)}x^2 {}_1F_1(1 + a^* - \nu; 2 - \nu; r)\right\}\right].$$

In order to construct a solution valid in the ideal and resistive regions, we must match the ideal solution for $|y| \to \infty$, (74) to the resistive solution, (92) for $|x| \to 0$ [17], [20]. If we define

$$y_0^2 = \frac{\langle B^2/|\nabla v|^2 \rangle}{\dot{q}^2 \langle B^2 \rangle} \frac{Q_0}{\alpha^2 \eta^*},$$ (93)

with

$$Q_0^3 = \frac{\dot{q}^2 \dot{\chi}^2 \langle B^2 \rangle \alpha^2 \eta^*}{\varrho \langle B^2/|\nabla v|^2 \rangle M},$$ (94)

we obtain from (58) and (59)

$$x^2 = (1/y_0^2 Q) y^2.$$ (95)

If we set

$$\Delta' \equiv a_2/a_1$$ (96)

(with a_1, a_2 from (74)) and

$$\Delta \equiv \frac{4 y_0^{1+2s} Q^{(5-2s)/4}}{\{Q^3 - (1+s-H)^2\}} \frac{\Gamma[\frac{1}{2}+s]}{\Gamma[-\frac{1}{2}-s]} \cdot \frac{\Gamma[\frac{1}{4}\{Q^{3/2}+3-2s-D_R/Q^{3/2}\}]}{\Gamma[\frac{1}{4}\{Q^{3/2}+1+2s-D_R/Q^{3/2}\}]},$$ (97)

then the condition for matching the resistive to the ideal solution is

$$\Delta = \Delta'.$$ (98)

In order to be able to carry through the matching for both positive and negative y values, the quantity Δ' must be constructed from the solution in the ideal region in such a way that it is the same for $y \to \infty$ and $y \to -\infty$. Starting with any two independent solutions $f_1(y)$, $f_2(y)$ of (72) (with $D = 0$), we determine the coefficients

$A_2^{(+)}{}^{(-)}_{2}$, defined by

$$\lim_{|y| \to \pm \infty} f_i = A_i^{(+),(-)} |y|^s + B_i^{(+),(-)} |y|^{-1-s},$$
$$i = 1, 2.$$ (99)

We then determine a factor k from the equation

$$\frac{B_1^{(+)} + k B_2^{(+)}}{A_1^{(+)} + k A_2^{(+)}} = \frac{B_1^{(-)} + k B_2^{(-)}}{A_1^{(-)} + k A_2^{(-)}}.$$ (100)

(Using the properties of the Wronskian of (72) (with $D = 0$), one can show that there are always real solutions k to (100).) With the so determined k one then constructs the solution

$$F(y) = f_1(y) + k f_2(y)$$ (101)

and determines a_1, a_2 (and thus Δ') from

$$\lim_{|y| \to \infty} F(y) = a_1 |y|^s + a_2 |y|^{-1-s}.$$ (102)

This yields

$$\frac{a_2}{a_1} = \frac{B_1^{(+)} + k B_2^{(+)}}{A_1^{(+)} + k A_2^{(+)}} = \frac{B_1^{(-)} + k B_2^{(-)}}{A_1^{(-)} + k A_2^{(-)}}.$$ (103)

IV. Discussion of the Dispersion Relation

When studying the dispersion relation (96)—(98), it ist necessary to keep in mind the restrictions imposed by the assumptions of small resistivity and growth rate, (69). With the definitions of y_0 and Q in (93) and (59) respectively, these conditions can be expressed as

$$1/y_0^2 \ll Q \ll y_0.$$ (104)

We now consider different cases, according to the sign of the driving term D_R.

IV.A) $D_R > 0$

When D_R is positive, the gamma functions in (97) have an infinite sequence of poles owing to the term $(-D_R)/4 Q^{3/2}$. Δ alternately vanishes and diverges, passing through all values many times. For a given Δ', there are infinitely many Q's which satisfy (98). Since y_0 scales as $(\eta^*)^{-1/3}$, the factor y_0^{1+2s} is large. Thus, the roots of (98) will in general be near the poles of $\Gamma[\frac{1}{4}\{Q^{3/2}+1+2s-D_R/Q^{3/2}\}]$, i.e.

$$Q_n^{3/2} = -(\tfrac{1}{2}+s+2n) + [(\tfrac{1}{2}+s+2n)^2 + D_R]^{1/2}.$$ (105)

The actual growth rate γ scales as $(\eta^*)^{1/3}$ since $\gamma \sim (\eta^*)^{1/3} Q$. This type of instability corresponds to the well known resistive interchanges [2].

IV.B) $D_R = 0$

In this case the term $D_R/4 Q^{3/2}$ no longer appears in the argument of the gamma functions and $\Gamma(a^*)$ has no poles. As can be seen from (92), it is required that $s < 1/2$ (for $s > 1/2$, the function F_0 does not have the correct form for matching to the ideal solutions, since $x^{1-2s} \gg x^0$ for $x \to 0$, and no unstable solution can be constructed unless $|\Delta'| \to \infty$).

Taking into account that $-1/2 < s < 1/2$, we can derive the following properties of Δ: when $0 \leq Q^3 \leq (1+s-H)^2$, Δ is positive and takes all values between 0 and $+\infty$. There is therefore always an instability when $\Delta' > 0$ since in this case (98) can always be satisfied (for small Q, Δ vanishes as $Q^{(5-2s)/4}$, Q is then proportional to $y_0^{(4+8s)/(2s-5)}$, which is consistent with condition (104), $1/y_0^2 \ll Q$.

Then, for small Q, the actual growth rate γ scales as

$$\gamma \sim (\eta^*)^{\frac{3+2s}{5-2s}}.$$

When $(1 + s - H)^2 \leq Q^3 \leq \infty$, Δ takes all values between $-\infty$ and 0, vanishing as $1/Q^{1+2s}$ for large Q. Taking into account that the results must be consistent with the condition (104) ($Q \ll y_0$), there is an instability only if $\Delta' \to -\infty$. (This is the ideal marginal limit. After careful examination it can be seen that this corresponds to subsequently making η^*/γ and γ small.)

IV.C) $D_R < 0$

This is the most interesting case since $D_R < 0$ stabilizes positive Δ'-values which are not too large.

First we observe, as in the case $D_R = 0$, that in general there are no instabilities if $s > 1/2$, i.e., unless $|\Delta'| \to \infty$ we can construct an unstable solutino only if the condition

$$(\tfrac{1}{2} - H)^2 + |D_R| < 1 \tag{106}$$

is satisfied.

For $Q^3 \geq (1 + s - H)^2$ we have the same situation as in the case $D_R = 0$, i.e. there are instabilities only when $\Delta' \to -\infty$ (ideal case).

As for $D_R = 0$, Δ is also positive for

$$Q^3 \leq (1 + s - H)^2.$$

The crucial difference arises for $Q \to 0$: when Q becomes small, the terms $|D_R|/4 Q^{3/2}$ in (97) become large and $\Delta \sim (y_0^2 Q)^{1/2+s}$.

When Δ' is positive, the resulting growth rates are so small that they are not consistent with the assumption (104) ($1/y_0^2 \ll Q$) unless Δ' is large.

We can estimate how large Δ' must be in order that the condition $Q y_0^2 \gg 1$ be satisfied: when Q decreases, the term $|D_R|/4 Q^{3/2}$ changes the behaviour $\Delta \sim Q^{(5-2s)/4}$ to $\Delta \sim Q^{1/2+s}$. Defining Q_c as the transition point at which this change of behaviour in $\Delta(Q)$ occurs, we obtain

$$\Delta_c = \Delta(Q_c). \tag{107}$$

If $\Delta' > \Delta_c$, the consistency condition $Q y_0^2 \gg 1$ is satisfied. Thus, in the range $0 \leq Q^3 \leq (1 + s - H)^2$ there is instability only if $\Delta' > \Delta_c$.

It is clear that the exact value of Q_c is somewhat arbitrary. However $|D_R|/4 Q_c^{3/2} \approx 1$ seems to be a good approximation, since the behaiour of

$$\frac{\Gamma[\tfrac{1}{4}\{Q^{3/2} + 3 - 2s + |D_R|/Q^{3/2}\}]}{\Gamma[\tfrac{1}{4}\{Q^{3/2} + 1 + 2s + |D_R|/Q^{3/2}\}]}$$

is already well represented by $(|D_R|/4 Q^{3/2})^{1/2-s}$ for values of $|D_R|/4 Q^{3/2}$ moderately larger than 1.

V. Conclusion

Resistive balloning modes in general three-dimensional configurations have been studied on the basis of the linearized equations of motion of resistive MHD.

On the assumption of small resistivity and perturbations localized transversally to the magnetic field, a stability criterion has been derived in the form of a coupled system of two second-order ordinary differential equations (Eqs. (27), (28)), without imposing any of the usual restrictions, e.g. symmetric configurations, circular plasma cross-section, large aspect ratio. The criterion obtained is rather general and contains as limiting cases the ideal balloning mode criterion and criteria for the stability of symmetric systems.

In the form of (27), (28) the stability criterion is, in general, rather implicit since its evaluation requires the solution of two coupled second-order ordinary differential equations on each closed field line, the coefficients depending both secularly and periodically on the independent variable along the field line.

Neglecting compressibility and assuming small growth rates (Sect. III. D), we obtain instability only when the condition (46) is satisfied, the growth rate scaling as $\gamma \sim (\eta^*)^{1/3} = (k_\perp^2 \eta)^{1/3}$. This contradicts the statement made in [15], according to which, under the same circumstances, an axisymmetric system is always unstable with growth rates proportional to η^*.

In Sect. III. F), (27), (28) have been considerably simplified by assuming small resistivity and inertia (conditions (69), (104)). Making use of two-length-scale expansion techniques, one obtains the averaged equation (84) and (85). Neglecting the effect of compressibility in these equations (i.e. assuming $G \gg 1$) decouples them. This leaves only one nontrivial equation, Eq. (86), which can be solved exactly. Satisfying the boundary conditions for this equation leads to a tearing-mode-like [2] dispersion relation for each closed field line

$$\Delta' = \Delta(Q, \eta^* \equiv k_\perp^2 \eta, H, D_R)$$

Eqs. (96)—(98), Δ' being determined from the asymptotic behaviour for large values of the inde-

pendent variable of the solution of the ideal, marginal-ballooning-mode equation. As in [2], there are always instabilities for $D_R > 0$. For $D_R = 0$ and Δ' finite there are no instabilities if $1/2 + s > 1$. If $1/2 + s < 1$ and $D_R = 0$, there are instabilities if either $\Delta' > 0$ or if $\Delta' \to -\infty$, the case $\Delta' \to -\infty$ corresponding to an ideally unstable configuration.

For $D_R < 0$ there are no instabilities if

$$\tfrac{1}{2} + s > 1 \,, \text{ unless } |\Delta'| \to \infty \,.$$

For $D_R < 0$, $1/2 + s < 1$ and $\Delta' > 0$, there are instabilities only if $\Delta' > \Delta_c'$ (Δ_c' given by (107)). For $\Delta' < 0$ there are only instabilities if $\Delta' \to -\infty$, i.e. the system is ideally unstable.

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